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Method of group foliation and non-invariant solutions of partial differential equations

Example: the heavenly equation

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Abstract. Using the heavenly equation as an example, we propose the method of group foliation as a tool for obtaining non-invariant solutions of PDEs with infinite-dimensional symmetry groups. The method involves the study of compatibility of the given equations with a differential constraint, which is automorphic under a specific symmetry subgroup and therefore selects exactly one orbit of solutions. By studying the integrability conditions of this automorphic system, *i.e.* the resolving equations, one can provide an explicit foliation of the entire solution manifold into separate orbits. The new important feature of the method is the extensive use of the operators of invariant differentiation for the derivation of the resolving equations and for obtaining their particular solutions. Applying this method we obtain exact analytical solutions of the heavenly equation, non-invariant under any subgroup of the symmetry group of the equation.

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1 Introduction

The general standard method for obtaining exact solutions of partial differential equations (PDEs) by symmetry analysis is symmetry reduction which gives only *invariant solutions*, *i.e.* solutions which are invariant with respect to some subgroup of the symmetry group of the PDE.

We are proposing the method of group foliation as a tool for obtaining non-invariant solutions of non-linear PDEs with infinite dimensional symmetry groups. The idea of the method, belonging to Lie and Vessiot [1,2], is more than a hundred years old being resurrected in a more modern form by Ovsiannikov 30 years ago (see [3] and references therein).

We have added to this method three important new ideas [4,5]: the use of *invariant cross-differentiation*, involving the operators of invariant differentiation and their commutator algebra, for the derivation of the resolving equations and for obtaining their particular solutions; the *commutator representation of the resolving system* in terms of the operators of invariant differentiation; the concept of *invariant integration* applied for solving the automorphic system.

In this paper on the example of the heavenly equation

$$u_{z\bar{z}} = \kappa(\mathrm{e}^u)_{tt} \quad \Longleftrightarrow \quad u_{xx} + u_{yy} = \kappa(\mathrm{e}^u)_{tt}, \qquad \kappa = \pm 1 \tag{1}$$

we clarify the main concepts of the method and consider in detail the main steps which should be performed for obtaining non-invariant solutions. This equation appears in the theory of gravitational instantons [6] where it describes self-dual Einstein spaces with Euclidean signature [7] having one rotational Killing vector.

2 Symmetry algebra

We start with finding the *symmetry algebra* of generators of the point transformations for the heavenly equation (1) [8]

$$T = \partial_t, \qquad G = t\partial_t + 2\partial_u,$$
$$X_a = a(z)\partial_z + \bar{a}(\bar{z})\partial_{\bar{z}} - (a'(z) + \bar{a}'(\bar{z}))\partial_u \tag{2}$$

where $a(z)$ is an arbitrary holomorphic functions of z and prime denotes derivative with respect to argument (see also [9]).

The Lie algebra of symmetry generators (2) is determined by the commutation relations

$$[T, G] = T, [T, X_a] = 0, [G, X_a] = 0, [X_a, X_b] = X_{ab'-ba'}. \tag{3}$$

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They show that the generators X_a of conformal transformations form a subalgebra of Lie algebra (3). This subalgebra is infinite dimensional since the generators X_a depend on $a(z)$. The corresponding finite transformations form an infinite dimensional symmetry subgroup of the equation (1) since instead of a group parameter they also involve an arbitrary holomorphic function of z.

We choose this infinite dimensional *conformal group* for the group foliation.

3 Differential invariants

Next we find differential invariants of the symmetry subgroup of conformal transformations. *Differential invariants* are the invariants of all the generators X_a in the *prolongation spaces*. This means that they can depend on independent variables, the unknowns and also on the partial derivatives of the unknowns allowed by the order of the prolongation. The *order* of the differential invariant is defined as the order of the highest derivative which this invariant depends on. The number N for the highest order invariant must be larger or equal to the order of the equation ($N \geq 2$) and must satisfy the requirement that there should be n functionally independent invariants with $n > p + q$ where p and q are the number of independent and dependent variables, respectively. In our case we have $p = 3,\ q = 1$ and $n > 4,\ N \geq 2$.

The routine calculation for $N = 2$ gives 5 functionally independent differential invariants up to the second order inclusively

$$t, \qquad u_t, \qquad u_{tt}, \qquad \rho = \mathrm{e}^{-u} u_{z\bar{z}}, \qquad \eta = \mathrm{e}^{-u} u_{zt} u_{\bar{z}t} \tag{4}$$

and all of them are real. This allows us to express the heavenly equation (1) solely in terms of the differential invariants

$$u_{tt} = \kappa\rho - u_t^2. \tag{5}$$

Thus, in our case we have $N = 2$ and $n = 5$ which is enough for the group foliation.

4 Automorphic system

Next we choose the general form of the automorphic system. We choose $p = 3$ invariants t, u_t, ρ as new *invariant independent variables*, the same number as in the original equation (1), and require that the *remaining invariants be functions of the chosen ones*. This provides us with the general form of the *automorphic system* that also contains the studied equation (5) expressed in terms of invariants (4)

$$\begin{cases} u_{tt} = \kappa\rho - u_t^2 \\ \ \eta = F(t, u_t, \rho). \end{cases} \tag{6}$$

The real function F in the right-hand side should be determined from the *resolving equations* which are compatibility conditions of the system (6). Then the system (6) will have the *automorphic property*, *i.e.* any of its solutions can be obtained from any other solution by an appropriate transformation of the conformal group.

5 Operators of invariant differentiation

Our next task is to find *operators of invariant differentiation*. They are linear combinations of the operators of total derivatives $D_t,\ D_z,\ D_{\bar{z}}$ with respect to independent variables $t, z, \bar{z}$

$$\delta = \lambda_1 D_t + \lambda_2 D_z + \lambda_3 D_{\bar{z}} = \sum_{i=1}^{3} \lambda_i D_i$$

with the coefficients λ_i which depend on local coordinates of the prolongation space. They are defined by the special property that, acting on any (differential) invariant, they map it again into a differential invariant. Being first order differential operators, they raise the order of a differential invariant by a unit. As a consequence, these differential operators commute with any infinitely prolonged generator $\overset{\infty}{X}_a$ of the conformal symmetry group. This implies the *determining equation* for the coefficients λ_i [3]

$$\overset{\infty}{X}_a(\lambda_i) = \sum_{j=1}^{3} \lambda_j D_j[\xi^i] \tag{7}$$

where $\xi^1 = \tau = 0,\ \xi^2 = \xi = a(z),\ \xi^3 = \bar{\xi} = \bar{a}(\bar{z})$ due to the definition (2) of X_a. It is obvious that the total number of independent operators of invariant differentiation is equal to the number of independent variables, *i.e.* 3 in our case.

Solving the equation (7) we obtain a *basis for the operators of invariant differentiation*

$$\delta = D_t, \qquad \Delta = \mathrm{e}^{-u} u_{\bar{z}t} D_z, \qquad \bar{\Delta} = \mathrm{e}^{-u} u_{zt} D_{\bar{z}}. \tag{8}$$

6 Basis of differential invariants

The next step is to find the *basis of differential invariants* which is defined as a minimal finite set of (differential) invariants of a symmetry group from which any other differential invariant of this group can be obtained by a finite number of invariant differentiations and operations of taking composite functions. The proof of the existence and finiteness of the basis was given by Tresse [10] and in a more modern form by Ovsiannikov [3].

In our example the basis of differential invariants is formed by the set of three invariants t, u_t, ρ, while two other invariants u_{tt} and η of equation (4) are given by the relations

$$u_{tt} = \delta(u_t), \qquad \eta \equiv \mathrm{e}^{-u} u_{zt} u_{\bar{z}t} = \Delta(u_t) = \bar{\Delta}(u_t). \tag{9}$$

All other functionally independent higher order invariants can be obtained by acting with operators of invariant differentiation on the *basis* $\{t, u_t, \rho\}$. In particular, the following third order invariants generated from the 2nd-order invariant ρ by invariant differentiations will be involved in our construction

$$\sigma = \Delta(\rho), \quad \bar{\sigma} = \bar{\Delta}(\rho), \quad \tau = \delta(\rho) \equiv \rho_t. \tag{10}$$

7 Commutator algebra of operators of invariant differentiation

The operators δ, Δ and $\bar{\Delta}$ defined by the formulas (8) form the *commutator algebra* which is a Lie algebra over the field of invariants of the conformal group [3].

This algebra is simplified by introducing two new operators of invariant differentiation Y and $\bar{Y}$ instead of Δ and $\bar{\Delta}$ and two new variables λ and $\bar{\lambda}$ instead of σ and $\bar{\sigma}$, defined by

$$\Delta = \eta Y, \quad \bar{\Delta} = \eta \bar{Y}, \quad \sigma = \eta\lambda, \quad \bar{\sigma} = \eta\bar{\lambda}. \tag{11}$$

The resulting algebra becomes

$$\begin{aligned} \left[\delta, Y\right] &= \left(\kappa\bar{\lambda} - 3u_t - \frac{\delta(\eta)}{\eta}\right) Y, \\ \left[\delta, \bar{Y}\right] &= \left(\kappa\lambda - 3u_t - \frac{\delta(\eta)}{\eta}\right) \bar{Y}, \\ \left[Y, \bar{Y}\right] &= \frac{(\tau + u_t\rho)}{\eta}\left(Y - \bar{Y}\right). \end{aligned} \tag{12}$$

With the use of operators δ, Y and $\bar{Y}$ the general form (6) of the automorphic system becomes

$$\begin{cases} \delta(u_t) = \kappa\rho - u_t^2 \\ Y(u_t) = 1 \qquad\qquad (\bar{Y}(u_t) = 1) \end{cases} \tag{13}$$

where the first equation is the heavenly equation and the second equation follows from the second relation (9). Here we put $\eta = F$ in the equations (11) and in the commutation relations (12) according to the 2nd equation in (6). Then we obtain $Y = (1/F)\Delta$ and $\bar{Y} = (1/F)\bar{\Delta}$. From the equation (10) we have

$$Y(\rho) = \lambda, \quad \bar{Y}(\rho) = \bar{\lambda}. \tag{14}$$

8 Derivation of resolving equations

The following step is to derive the *resolving equations*. This is a set of compatibility conditions between the studied equation and those that we have added to obtain the automorphic system. In our case we require compatibility between the two equations (13) which gives restrictions on the function $F(t, u_t, \rho)$ in the right-hand side of the second equation in (6). A new feature in our modification of the method is that we do this in an explicitly invariant manner by using the *invariant cross-differentiation* [4,5] involving the operators δ, Y and $\bar{Y}$.

We start with the integrability condition for the system (13) which we obtain by the invariant cross-differentiation with δ and Y using their commutation relation (12)

$$\delta(F) = \left[\kappa(\lambda + \bar{\lambda}) - 5u_t\right]F. \tag{15}$$

The definitions of $\lambda, \bar{\lambda}$ which appear here are given by two equations (14). The compatibility condition for the equations (14) is obtained by the invariant cross-differentiation with $\bar{Y}$ and Y using their commutation relation (12)

$$F\big(Y(\bar{\lambda}) - \bar{Y}(\lambda)\big) = (\tau + u_t\rho)(\lambda - \bar{\lambda}). \tag{16}$$

The definition of τ which appear here is given in the equation (10)

$$\delta(\rho) = \tau. \tag{17}$$

Using invariant cross-differentiations with δ and Y or $\bar{Y}$, we obtain two compatibility conditions of equation (17) with each of the two equations (14)

$$\delta(\lambda) = Y(\tau) + 2u_t\lambda - \kappa\lambda^2, \tag{18}$$

$$\delta(\bar{\lambda}) = \bar{Y}(\tau) + 2u_t\bar{\lambda} - \kappa\bar{\lambda}^2. \tag{19}$$

In a similar way we obtain the last resolving equation

$$\begin{aligned} &F\big(Y(\bar{\lambda}) + \bar{Y}(\lambda)\big) = -(\tau + u_t\rho)(\lambda + \bar{\lambda}) \\ &+2\kappa\left[\delta(\tau) + 4u_t\tau + 2F + \kappa\rho^2 + 2u_t^2\rho\right] \end{aligned} \tag{20}$$

where no new differential invariants appear.

The resolving equations (15, 16, 18, 19) and (20) form a closed *resolving system* where the 2nd-order differential invariant $\eta = F$ and the 3rd-order differential invariants λ, $\bar{\lambda}$ and τ are functions of t, u_t, ρ. They should be regarded as additional unknowns in these equations, so the resolving system consists of 5 partial differential equations with 4 unknowns $F, \lambda, \bar{\lambda}$ and τ and 3 independent variables t, u_t, ρ.

Next we consider the operators of invariant differentiation projected on the solution manifold of the heavenly equation and on the space of differential invariants treated as new independent variables

$$\delta = \partial_t + (\kappa\rho - u_t^2)\partial_{u_t} + \tau\partial_\rho, \; Y = \partial_{u_t} + \lambda\partial_\rho, \; \bar{Y} = \partial_{u_t} + \bar{\lambda}\partial_\rho. \tag{21}$$

When we use these expressions in the resolving equations (15, 16, 18, 19) and (20), we obtain an explicit form of the resolving system.

The commutator relations (12) were satisfied identically by the operators of invariant differentiation. On the contrary, for the *projected operators* (21) these commutation relations and even the Jacobi identity

$$\left[\delta, [Y, \bar{Y}]\right] + \left[Y, [\bar{Y}, \delta]\right] + \left[\bar{Y}, [\delta, Y]\right] = 0 \tag{22}$$

are not satisfied identically but only on account of the resolving equations. It is easy to check that even a stronger statement is valid.

Theorem 1 *The commutator algebra* (12) *of the operators of invariant differentiation* $\delta, Y, \bar{Y}$, *together with the Jacobi identity* (22), *is equivalent to the resolving system for the heavenly equation and hence provides a commutator representation for this system.*

This theorem means that the complete set of the resolving equations is encoded in the commutator algebra of the operators of invariant differentiation and provides the easiest way to derive the resolving system [4,5]. We shall see that the commutator representation of the resolving system can lead to a useful ansatz for finding a particular solution of this system.

9 Particular solutions of resolving system

To find particular solutions of the resolving system, we make various simplifying assumptions. The most obvious ones, like $\bar{Y} = Y$ or $F = 0$, lead to invariant solutions. These we already know, or can obtain by much simpler standard methods.

The weaker assumption that leads to non-invariant solutions is that the operators Y and $\bar{Y}$ commute

$$[Y, \bar{Y}] = 0 \quad \Longleftrightarrow \quad \tau = -u_t\rho \tag{23}$$

but $\bar{Y} \neq Y$, *i.e.* $\bar{\lambda} \neq \lambda$ and also $F \neq 0$. With this ansatz we find the *particular solution of the resolving system* [5]

$$F = \rho^3\varphi(\xi, \theta),\ \tau = -u_t\rho,\ \lambda = \kappa u_t + \mathrm{i}\sqrt{2\kappa\rho - u_t^2} \tag{24}$$

and $\bar{\lambda}$ has the opposite sign before the square root. Here the condition $2\kappa\rho - u_t^2 \geq 0$ is imposed, φ is an arbitrary real smooth function and

$$\xi = \frac{2\kappa\rho - u_t^2}{\rho^2}, \qquad \theta = t - \frac{\kappa}{\rho}\left(u_t + \sqrt{2\kappa\rho - u_t^2}\right).$$

10 Reconstruction of non-invariant solutions of heavenly equation

To reconstruct solutions of the heavenly equation starting from the particular solution (24) of the resolving system we use the procedure of *invariant integration* which amounts to the transformation of equations to the form of *exact invariant derivative* [5]. Then we drop the operator of invariant differentiation in such an equation adding the term that is an *arbitrary element of the kernel* of this operator. This term plays the role of the integration constant.

To be explicit, we start from our ansatz (23) in the form $D_t(\ln\rho) = D_t(-u)$. We integrate this equation: $\ln\rho = -u + \ln\gamma_{z\bar{z}}(z, \bar{z})$ where the last term is a function to be determined. This gives $\rho = \mathrm{e}^{-u}u_{z\bar{z}} = \mathrm{e}^{-u}\gamma_{z\bar{z}}(z, \bar{z})$ and hence $u_{z\bar{z}} = \gamma_{z\bar{z}}(z, \bar{z})$. This implies the following form of solutions

$$u(t, z, \bar{z}) = \gamma(z, \bar{z}) + \alpha(t, z) + \bar{\alpha}(t, \bar{z}) \tag{25}$$

where γ, α and $\bar{\alpha}$ are arbitrary smooth functions of two variables.

Next we rewrite the formulas (24) for λ and $\bar{\lambda}$ in the form of *exact invariant derivatives*

$$Y\left(\sqrt{2\kappa\rho - u_t^2} - \mathrm{i}\kappa u_t\right) = 0,\ \bar{Y}\left(\sqrt{2\kappa\rho - u_t^2} + \mathrm{i}\kappa u_t\right) = 0.$$

These equations are integrated in the form

$$\sqrt{2\kappa\rho - u_t^2} + \mathrm{i}\kappa u_t = \psi(t, z),\ \sqrt{2\kappa\rho - u_t^2} - \mathrm{i}\kappa u_t = \bar{\psi}(t, \bar{z})$$

where ψ is an arbitrary smooth function and $\bar{\psi}$ is complex conjugate to ψ.

We skip further details and present only the *final result* [5] (see also [11] in relation to (27)).

1. *Solution of the heavenly equation* $u_{z\bar{z}} = (\mathrm{e}^u)_{tt}$:

$$u(t, z, \bar{z}) = \ln\left|\frac{\big(t + b(z)\big)c'(z)}{c(z) + \bar{c}(\bar{z})}\right|^2. \tag{26}$$

2. *Solution of the heavenly equation* $u_{z\bar{z}} = -(\mathrm{e}^u)_{tt}$:

$$u(t, z, \bar{z}) = \ln\left|\frac{\big(t + b(z)\big)c'(z)}{1 + |c(z)|^2}\right|^2. \tag{27}$$

Here $b(z)$ and $c(z)$ are arbitrary holomorphic functions. One of them is fundamental and the choice of it corresponds to a particular *orbit of solutions*. The other one is induced by a conformal symmetry transformation and can be transformed away. These solutions for generic $b(z)$ and $c(z)$ are *non-invariant solutions* of the heavenly equation.

11 Conclusions and outlook

We conclude that, unlike the method of symmetry reduction, group foliation can be applied for constructing non-invariant solutions of PDEs. A regular approach for solving the resolving equations in terms of invariant derivatives is now in progress. In [4] we constructed the group foliation of the complex Monge-Ampère equation. We hope to obtain its non-invariant solutions generating the metric with no Killing vectors for the gravitational instanton $K3$.

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